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The spin glass in an AC field

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Abstract. We consider the dynamical theory for spin glasses developed by Sompolinsky, Hertz and others. The additional relations between the freezing parameters and the anomalous response parameters in their theory are considered. We compare the dynamical theory with the probability distribution method of the overlap of the magnetisation between two different states. Finally we discuss the susceptibility of the spin glass in an AC field with a small finite amplitude, and the results explain the experimental results qualitatively.

1. Introduction

We concentrate on the model with an infinitely long range random interaction to study the properties of spin glasses, and we call it the SK model (Sherrington and Kirkpatrick 1975). The Hamiltonian is given by

$$H_{\mathrm{SK}} = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i \qquad \sigma_i = \pm 1 \tag{1}$$

where the sum $\Sigma_{\langle ij\rangle}$ runs over all pairs of spins, and the J_{ij} are random interactions obeying the probability distribution

$$P(J_{ij}) = (N/2\pi J^2)^{1/2}\exp(-J_{ij}^2 N/2J^2) \tag{2}$$

where N is the number of spins.

The SK model has been studied by the replica method in detail (Edwards and Anderson 1975). First Sherrington and Kirkpatrick (1975) presented the replica symmetric solution (we call it the SK solution), but this solution was proved to be unstable in the spin glass phase (de Almeida and Thouless 1978). In order to obtain the stable solution symmetry breaking in the replica space is required. It is believed that the Parisi replica symmetry breaking scheme (Parisi 1980a, b) gives the stable solution of the SK model (Thouless *et al* 1980, De Dominicis and Kondor 1983). We call it the Parisi solution. The order parameters in the Parisi solution are expressed by the function $q(x)$ where the parameter x is changeable from 0 to 1. But within this theory we have some questions: what is the physical meaning of the replica symmetry breaking, and what is the physical meaning of the parameter x? In order to study these questions, two main approaches have been used. The one is the dynamical theory of the SK model which has been developed by Sompolinsky (1981), Hertz (1983a, b) and others. The other is the approach by studying the probability distribution of the overlap of the magnetisation between two different states (Parisi 1983, De Dominicis and Young 1983). These approaches are based on the concept which says that there are many metastable states in the spin glass phase (Bray and Moore 1980, 1981, De Dominicis *et al* 1980), but the relationship between these approaches is not clear. We try to consider the relation between these approaches in this paper.

In this paper we consider the dynamical theory of the SK model. We mainly follow the Hertz method (Hertz 1983a, b), but it was difficult to determine susceptibilities in his theory. Therefore in § 2, we present a model with no obscurity and review the Sompolinsky and Hertz discussions. The dynamical theory gives the equations for the relationship between freezing parameters (Edwards and Anderson 1975) and anomalous response parameters (Sommers 1978). In order to determine these parameters, other additional relation equations are required. In § 3, we consider the additional relationship equations (Sompolinsky (1981) presented those which lead to the Parisi solution, but had no further consideration of it, and Hertz (1983b) discussed it, but I could not understand Hertz's discussion). In § 4, we consider the Parisi probability distribution $\bar{P}(q)$ of the overlap q from the viewpoint of dynamical theory. In § 5, we discuss the susceptibilities in the spin glass, especially the AC susceptibility. It is discussed how a static uniform field can affect the static freezing parameter $q(x=0)$ and an AC uniform field with a long cycle time can affect the dynamical freezing parameter $q(x=1)$ which connects with the anomalous response parameter through the fluctuation-dissipation theorem (FDT). In a previous paper (Shirakura 1984a), we discussed the susceptibilities of the spin glass in a static small field. In this section we discuss the susceptibility of the spin glass in an AC field with a small but finite amplitude. Section 6 presents the discussion of the results.

2. The Sompolinsky and Hertz dynamical theory in the spin glass

The SK model (equations (1) and (2)) has Ising spins $\sigma_i = \pm 1$. Here we consider that the system has Langevin-type dynamics. Therefore we add the weight part $W(u)$ to the Hamiltonian

$$\hat{H} = H_{\mathrm{SK}} + W(u) \tag{3}$$

$$W(u) = \tfrac{1}{8} u \sum_i (\sigma_i^2 - 1)^2 \tag{4}$$

and we consider the spins to be changeable from $-\infty$ to ∞ continuously. If we take the limit $u \to \infty$, we have Ising spins such as

$$\lim_{u\to\infty} \exp[-\beta W(u)] \propto \prod_i [\delta(\sigma_i - 1) + \delta(\sigma_i + 1)]. \tag{5}$$

When $u \neq \infty$, we put a constraint $\Sigma_i \sigma_i^2 = N$. This constraint removes the obscurity of the determination of susceptibilities in the Hertz theory. The partition function with this constraint is given by

$$Z = \int_{-\infty}^{\infty} \prod_i \mathrm{d}\sigma_i \delta\left(N - \sum_i \sigma_i^2\right) \exp(-\beta \hat{H})$$

$$= \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\beta\,\mathrm{d}y}{2\pi \mathrm{i}} \exp(N\beta y - \tfrac{1}{8}\beta u N) \int_{-\infty}^{\infty} \prod_i \mathrm{d}\sigma_i \exp(-\beta H_{\mathrm{eff}}) \tag{6}$$

where

$$H_{\mathrm{eff}} \equiv -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j + \sum_i (\tfrac{1}{2} r \sigma_i^2 + \tfrac{1}{8} u \sigma_i^4 - h_i \sigma_i) \tag{7}$$

$$r \equiv 2y - \tfrac{1}{2}u.$$

The y integral can be found by steepest descents in the thermodynamic limit $N\to\infty$. The saddle point equation is given by

$$\sum_i \langle\sigma_i^2\rangle_{\text{eff}} = N \tag{8}$$

where

$$\langle\ldots\rangle_{\text{eff}} \equiv \int_{-\infty}^{\infty}\prod_i \mathrm{d}\sigma_i\ldots\exp(-\beta H_{\text{eff}})\left(\int_{-\infty}^{\infty}\prod_i \mathrm{d}\sigma_i\exp(-\beta H_{\text{eff}})\right)^{-1}. \tag{9}$$

Let $P(\{\sigma_i\}, t)\,\Pi_i\,\mathrm{d}\sigma_i$ be the probability of finding the system in the area $\Pi_i\,\mathrm{d}\sigma_i$ at $\{\sigma_i\}$ and at time t. We consider the simplest Langevin-type equation of motion to be $P(\{\sigma_i\}, t)\propto\exp(-\beta H_{\text{eff}})$ for a stationary probability distribution, as follows:

$$\mathrm{d}\sigma_i/\mathrm{d}t = -\gamma_0\,\mathrm{d}H_{\text{eff}}/\mathrm{d}t + \eta_i(t) \tag{10}$$

$$\langle\eta_i(t)\eta_j(t')\rangle = 2T\gamma_0\delta_{ij}\delta(t-t'). \tag{11}$$

The r in H_{eff} is determined by (8), as follows:

$$[\langle\sigma_i^2\rangle]_{\text{a}} = 1 \tag{12}$$

where $[\ldots]_{\text{a}}$ denotes the random configurational average, and the Langevin noise average is denoted by $\langle\ldots\rangle$.

We concentrate on the response function $G_{ij}(\omega)$:

$$[\langle\sigma_i(\omega)\rangle]_{\text{a}} = \sum_j G_{ij}(\omega)\hat{h}_j(\omega) + (\text{other terms except for the first order terms of } \hat{h}) \tag{13}$$

and the correlation functions

$$C_{ij}(\omega)2\pi\delta(\omega+\omega') = [\langle\sigma_i(\omega)\sigma_j(\omega')]_{\text{a}} - [\langle\sigma_i(\omega)\rangle\langle\sigma_j(\omega')\rangle]_{\text{a}} \tag{14a}$$

$$\hat{C}_{ij}(\omega)2\pi\delta(\omega+\omega') = [\langle\sigma_i(\omega)\sigma_j(\omega')\rangle]_{\text{a}} \tag{14b}$$

where we write

$$h_i = \hat{h}_i + H. \tag{15}$$

$\hat{h}_i$ is an infinitesimally small field to observe a response and H is a uniform external field. When $H=0$, G_{ij} with $i\neq j$ is zero. When $H\neq 0$, G_{ij} with $i\neq j$ has the magnitude $\mathrm{O}((1/N)^1, H^2)$. Therefore we concentrate on the diagonal terms G_{ii}, C_{ii} and $\hat{C}_{ii}$ only. For simplicity we drop the subscripts, i.e. $G=G_{ii}$, $C=C_{ii}$ and $\hat{C}=\hat{C}_{ii}$. The G and C are related by the fluctuation-dissipation theorem (FDT) (Ma 1976)

$$C(\omega) = (2T/\omega)\,\mathrm{Im}\,G(\omega). \tag{16}$$

Here we consider the approximation discussed by Hertz (1983a, b). We consider $H=0$, and $G(\omega)$ and $\hat{C}(\omega)$ are calculated from the equation of motion (10) at the self-consistent two-loop level for the term $\frac{1}{8}u\sigma_i^4$ and to lowest order in $1/N$, as follows:

$$G^{-1}(\omega) = G_0^{-1}(\omega) + \sum(\omega) = r - \mathrm{i}\omega/\gamma_0 + \sum(\omega) \tag{17}$$

$$\hat{C}(\omega) = G(\omega)\hat{\Lambda}(\omega)G(-\omega) \tag{18}$$

$$\sum(\omega) = -J^2G(\omega) + \tfrac{3}{2}u\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\hat{C}(\omega') - (9u^2/2)\int_{-\infty}^{\infty}(\mathrm{d}\omega'\,\mathrm{d}\omega''/(2\pi)^2)\hat{C}(\omega')\hat{C}(\omega'')G(\omega-\omega'-\omega'') \tag{19}$$

$$\hat{\Lambda}(\omega) = 2T/\gamma_0 + J^2\hat{C}(\omega) + (3u^2/2)\int_{-\infty}^{\infty}(\mathrm{d}\omega'\,\mathrm{d}\omega''/(2\pi)^2)\hat{C}(\omega')\hat{C}(\omega'')\hat{C}(\omega-\omega'-\omega''). \tag{20}$$

First we consider the SK solution in this model. In the spin glass (SG) phase the correlation function $\hat{C}(t)$ has a time persistent part:

$$\hat{C}(t)=\bar{C}(t)+q \qquad \lim_{t\to\infty}\bar{C}(t)=0. \tag{21}$$

We call the magnitude of the time persistent part of $\hat{C}(t)$ a freezing parameter. With the Fourier transform of (21), we have

$$\hat{C}(\omega)=\bar{C}(\omega)+q2\pi\delta(\omega). \tag{22}$$

We consider that $G(\omega)$ and $\bar{C}(\omega)$ are related by the FDT ($\bar{C}(\omega)=(2T/\omega)\mathrm{Im}\,G(\omega)$). The static response $G(\omega=0)$ is calculated from the FDT, the Kramers-Kronig relation and equations (12) and (21), as follows:

$$G(\omega=0)=(1/T)\bar{C}(t=0)=(1/T)(\hat{C}(t=0)-q)=(1/T)(1-q). \tag{23}$$

The determination equation for q is given by picking out the part proportional to the $\delta(\omega)$ function in $\hat{C}=G(\omega)\hat{\Lambda}(\omega)G(-\omega)$, as follows:

$$q=G^2(\omega=0)[J^2q+(3u^2/2)q^3]. \tag{24}$$

But it is shown (Hertz 1984a) that this solution is unstable in the SG phase. In the SG phase we have the negative effective kinetic coefficient $\gamma(\omega=0)<0$ which shows the instability of this solution, where $\gamma(\omega)$ is defined by

$$\gamma^{-1}(\omega)\equiv(G^{-1}(-\omega)-G^{-1}(\omega))/2\mathrm{i}\omega. \tag{25}$$

To obtain a stable solution Hertz (1983a) presented the next solution (we call it the Hertz solution). First, instead of (22), the correlation function $\hat{C}(\omega)$ is rewritten as

$$\hat{C}(\omega)=\bar{C}(\omega)+2q\varepsilon(\omega^2+\varepsilon^2)^{-1} \qquad \varepsilon\ll\gamma_0 T \tag{26}$$

in a finite system, where the δ function in $\hat{C}$ gets smeared out. The q decays with a relaxation time $\tau\simeq\varepsilon^{-1}$. We consider that $G(\omega)$ and $\hat{C}(\omega)$ are related by the FDT ($\hat{C}(\omega)=(2T/\omega)\,\mathrm{Im}\,G(\omega)$). Then we have

$$G(\omega)=\bar{G}(\omega)+\hat{\delta}\varepsilon(-\mathrm{i}\omega+\varepsilon)^{-1} \tag{27}$$

$$\hat{\delta}=q/T \qquad \bar{C}(\omega)=(2T/\omega)\,\mathrm{Im}\,\bar{G}(\omega). \tag{28}$$

The second term in the right-hand side in (27) is an anomalously slow-time response term, so we call the $\hat{\delta}$ an anomalous response parameter. In the Hertz solution, q and $\hat{\delta}$ are related by the FDT (equation (28)). In the SK solution we have $\hat{\delta}=0$. In a similar way to deriving (23), we have

$$\bar{G}(\omega=0)=(1-q)/T. \tag{29}$$

The determination equation for q is given by integrating $\hat{C}(\omega)=G(\omega)\hat{\Lambda}(\omega)G(-\omega)$ over ω between $-\omega_1$ and ω_1, where $\gamma_0 T\gg\omega_1\gg\varepsilon$, and we have

$$q=(1-q+q^2/2)[J^2q+(3u^2/2)q^3]/T^2. \tag{30}$$

This solution is stable for a short timescale $t\ll\varepsilon^{-1}$ in the SG phase (Hertz 1983a). Hertz discussed the possibility that this solution describes the short time properties (the ZFC susceptibility, etc) of the spin glass better.

The solution which is stable for a long timescale in the SG phase was first discussed in the dynamical theory by Sompolinsky (1981). He assumed that in a finite system the decay of the freezing parameter occurs in a distribution of many large relaxation times (all of which become infinite in the thermodynamic limit). In this situation $\hat{C}(\omega)$ is written as

$$\hat{C}(\omega)=\bar{C}(\omega)+\sum_{j=1}^{\hat{N}} q_j[2\varepsilon_j(\omega^2+\varepsilon_j^2)^{-1}] \tag{31}$$

where we assume $\varepsilon_1^{-1}\gg\varepsilon_2^{-1}\gg\ldots\gg\varepsilon_{\hat{N}}^{-1}$. q_1 is the magnitude of freezing which comes from the largest spin cluster with the longest relaxation time and $q_{\hat{N}}$ is the magnitude of freezing which comes from the smallest spin clusters with the shortest relaxation time. The anomalous response term in $G(\omega)$ is similarly divided into $\hat{N}$ parts, as follows:

$$G(\omega)=\bar{G}(\omega)+\sum_{j=1}^{\hat{N}}\hat{\delta}_j\varepsilon_j(-\mathrm{i}\omega+\varepsilon_j)^{-1}. \tag{32}$$

We assume that the finite time parts $\bar{G}(\omega)$ and $\bar{C}(\omega)$ in the thermodynamic limit $N\to\infty$ are related by the FDT. In a similar way for deriving (23) and (29), we have

$$\bar{G}(\omega=0)=\left(1-\sum_{j=1}^{\hat{N}}q_j\right)\Big/T. \tag{33}$$

A set of relations between $\{q_j\}$ and $\{\hat{\delta}_j\}$ is obtained from integrating $\hat{C}(\omega)=G(\omega)\hat{\Lambda}(\omega)G(-\omega)$ over ω between $-\omega_j$ and ω_j, where $\varepsilon_j\ll\omega_j\ll\varepsilon_{j+1}$, $j=1,\ldots,\hat{N}$, as follows:

$$\begin{aligned}q_n=&\left[\left(G(0)-\sum_{j=1}^{n}\hat{\delta}_j\right)\left(G(0)-\sum_{j=1}^{n-1}\hat{\delta}\right)+\hat{\delta}_n^2/2\right]\\&\times\left[J^2q_n+\left(\frac{3u^2}{2}\right)q_n^3+\left(\frac{9u^2}{2}\right)\left(\sum_{j=1}^{n-1}q_j\right)q_n^2+(9u^2/2)\left(\sum_{j=1}^{n-1}q_j\right)^2q_n\right]\\&n=1,\ldots,\hat{N}\end{aligned} \tag{34}$$

$$G(0)=\left(1-\sum_{j=1}^{\hat{N}}q_j\right)\Big/T+\sum_{j=1}^{\hat{N}}\hat{\delta}_j. \tag{35}$$

Another set of relations between $\{q_j\}$ and $\{\hat{\delta}_j\}$ is required to obtain solutions for $\{q_j\}$ and $\{\hat{\delta}_j\}$. If we assume

$$T\hat{\delta}_j=(j/\hat{N})q_j \qquad j=1,\ldots,\hat{N} \tag{36}$$

and take the continuum limit $\hat{N}\to\infty$

$$n/\hat{N}=x_n \qquad \sum_{j=1}^{n}q_j=q(x_n) \tag{37}$$

we have the Parisi solution (Sompolinsky 1981, Hertz 1983b):

$$\mathrm{d}q(x)/\mathrm{d}x=[G(0)-\hat{\delta}(x)]^2[J^2+(9u^2/2)q^2(x)]\,\mathrm{d}q(x)/\mathrm{d}x \tag{34$'$}$$

$$G(0)=(1-q(1))/T+\hat{\delta}(1) \tag{35$'$}$$

$$T\hat{\delta}(x)=\int_0^x[q(x)-q(x')]\,\mathrm{d}x'. \tag{36$'$}$$

In the next section we consider the relations (36).

3. The consideration for the additional relations between freezing parameters and anomalous response parameters

In this section we consider the additional relations between freezing parameters and anomalous response parameters. In a finite system we write a static susceptibility as follows:

$$[\delta\langle\sigma_i\rangle/\delta h]_{\mathrm{a}}|_{h=0} \tag{38}$$

where δh is a uniform static field which becomes zero in the thermodynamic limit $N\to\infty$.

First we discuss the case of pure ferromagnets. In the ferromagnetic phase the free energy plotted against magnetisation $m=\langle\sigma_i\rangle$ is shown in figure 1. It has two minimum states. In a field δh, the probability to stay in one state depends on the Boltzmann factor

$$\exp\left(-\beta\delta h\sum_i\langle\sigma_i\rangle\right). \tag{39}$$

In the ferromagnetic phase we have $\Sigma_i\langle\sigma_i\rangle\simeq O(N)$. Therefore if we take $\delta h\simeq O(T/N)$, the mixing between two minimum states in a long timescale is forbidden. We consider that the anomalous response comes from mixing between states. Therefore the anomalous response should not be included in the ferromagnetic mean-field theory.

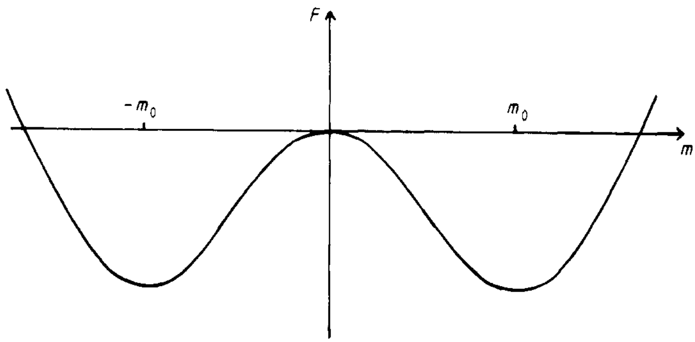

Figure 1. The free energy plotted against magnetisation $m=\langle\sigma_i\rangle$ in the ferromagnetic phase.

Next we discuss the case of a spherical spin glass (Kosterlitz *et al* 1976, Nemoto and Takayama 1984). The model in § 2 expresses the spherical spin glass when $u=0$. In the spherical spin glass the replica symmetry breaking (RSB) does not occur and the SK solution is stable at all temperatures. We can discuss the spherical spin glass in a similar way to the case of pure ferromagnets. We can consider that there is one spin cluster over a whole system in the SG phase of the spherical SG, and the magnitude of its total magnetisation is of the order of $N^{1/2}$ on average, i.e. $\Sigma_i\langle\sigma_i\rangle\simeq O(N^{1/2})$. Therefore if we take $\delta h\simeq O(T/N^{1/2})$, the mixing between minimum states in a long timescale is forbidden. So we do not have the anomalous response (we can see that (34) and (35) with $u=0$ are the exact equation of state for the spherical SG (Kosterlitz *et al* 1976) if we take q_1 non-zero and all other parameters zero).

Finally we discuss the case of the SG with $u\neq 0$. The situation is quite different from the spherical SG. In the present case there is a constraint on the magnitude of each spin at each site. This constraint causes the frustration effect. Therefore we have a distribution of various sizes of spin clusters. We distinguish between these spin clusters by its relaxation times $\varepsilon_1^{-1} \gg \varepsilon_2^{-1} \gg \ldots \gg \varepsilon_{\hat{N}}^{-1}$. We consider that the spin cluster with a relaxation time ε_n^{-1} has magnetisation m_n on average. The probability for the mixing between an up state m_n and a down state $-m_n$ of the spin cluster with a relaxation time ε_n^{-1} in a long timescale is given by

$$\omega_n = \exp(-\beta\delta h m_n)[\exp(-\beta\delta h m_n)+\exp(\beta\delta h m_n)]^{-1} \qquad n=1,\ldots,\hat{N} \tag{40}$$

in a field δh. We consider several trial considerations.

(i) If we could have a field δh which forbids any flip of any spin cluster, we do not have any anomalous response $\hat{\delta}_j = 0, j=1,\ldots,\hat{N}$. In this case we have $\mathrm{d}q(x)/\mathrm{d}x = 0$ in (34′), (35′) and (36′), and this solution is reduced to the SK solution.

(ii) If any field δh (which becomes zero in the limit $N \to \infty$) could not forbid any flip of any spin clusters, freezing parameters and anomalous response parameters are related by the FDT, i.e. $T\hat{\delta}_j = q_j, j=1,\ldots,\hat{N}$. But in this case we also have $\mathrm{d}q(x)/\mathrm{d}x = 0$ in (34′)-(36′), and this solution is reduced to the Hertz solution.

Here we consider that the largest spin cluster with the longest relaxation time ε_1^{-1} has magnetisation $m_1 \simeq O(N^{1/2})$ similar to the case of a spherical SG, and the smallest spin clusters with the shortest relaxation time $\varepsilon_{\hat{N}}^{-1}$ have magnetisation $m_{\hat{N}} \simeq O(1)$, i.e.

$$\begin{aligned} m_1 &\simeq O(N^{1/2}) \\ &\vdots \\ m_{\hat{N}} &\simeq O(1). \end{aligned} \tag{41}$$

We take $\delta h \simeq O(T/N^{1/2})$ similar to the case of the spherical SG. Then the δh forbids the flip of the largest spin cluster, $T\hat{\delta}_1 = 0$. On the contrary, the smallest spin clusters have flips freely in a long timescale, and we have $T\hat{\delta}_{\hat{N}} = q_{\hat{N}}$ by the FDT.

(iii) If we assume $T\hat{\delta}_j = 0$ for $1 \le j \le n$ and $T\hat{\delta}_j = q_j$ for $n+1 \le j \le \hat{N}$, this solution is reduced to the solution with two parameters ($T\hat{\delta}_1 = 0$, $T\hat{\delta}_{\hat{N}} = q_{\hat{N}}$) only. It is shown that this solution corresponds to the solution presented by Sommers (1978). We call it the Sommers solution.

(iv) The Parisi solution is obtained by connecting linearly between $T\hat{\delta}_1 = 0$ and $T\hat{\delta}_{\hat{N}} = q_{\hat{N}}$, $T\hat{\delta}_j = (j/\hat{N})q_j$, $j=1,\ldots,\hat{N}$.

These discussions suggest that the Parisi parameter $x_j = j/\hat{N}$ may be related to the probability of the mixing between an up state m_j and a down state $-m_j$ of the spin cluster with a relaxation time ε_j^{-1} in a long timescale in a field $\delta h \simeq O(T/N^{1/2})$.

4. Considerations for relations between the dynamical method and the probability distribution method of the overlap of magnetisation

In this section we consider the probability distribution $\bar{P}(q)$ of the overlap of the magnetisation between two different states from the standpoint of dynamical theory.

First we simply review the idea presented by Parisi (1983) and De Dominicis and Young (1983). From the suggestions of the computer simulation for the SK model (Mackenzie and Young 1983) we have the simple picture of the free energy structure plotted against phase space in the SG phase such as shown in figure 2. As shown in figure 2, we have a barrier with a height $O(N^{1/2})$ which separates states with the same sign of total magnetisation and other states with the inverse total magnetisation. We assume that the mixing between any former state and any latter one never occurs in physical observation times. Therefore we concentrate on the states on one side where

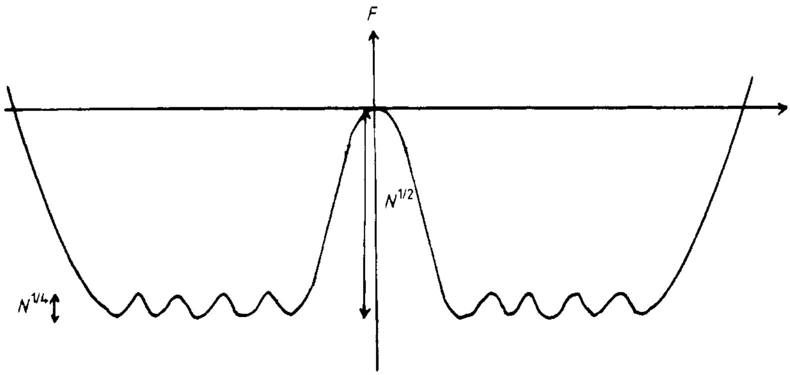

Figure 2. The free energy plotted against phase space in the spin glass phase which is suggested by the computer simulations (Mackenzie and Young 1983).

there are barriers with height $O(N^{1/4})$. We assume that the states on one side can be mixed in a long timescale. We consider that the state α occupies the phase space Ω_α and has local magnetisation $\{m_i^\alpha\}$. In a short timescale, we have a magnitude of freezing

$$q_{EA}=\left[\sum_\alpha p_\alpha(m_i^\alpha)^2\right]_{\mathrm{a}} \tag{42}$$

where $p_\alpha=\mathrm{Tr}_{\Omega_\alpha}\exp(-\beta H)/\Pi_\alpha\,\mathrm{Tr}_{\Omega_\alpha}\exp(-\beta H)$. In a long timescale, we have a magnitude of freezing

$$\bar{q}=\left[\left(\sum_\alpha p_\alpha m_i^\alpha\right)^2\right]_{\mathrm{a}}. \tag{43}$$

Here we consider the probability distribution $\bar{P}(q)$ of the overlap of the magnetisation between two different states defined by

$$\bar{P}(q)=\left(\sum_{\alpha,\beta} p_\alpha p_\beta\delta(q-q_{\alpha\beta})\right)_a \tag{44}$$

$$q_{\alpha\beta}=(1/N)\sum_i m_i^\alpha m_i^\beta. \tag{45}$$

Using the $\bar{P}(q)$, $\bar{q}$ is written as

$$\bar{q}=\int_0^1 q\bar{P}(q)\,\mathrm{d}q. \tag{46}$$

It is considered (De Dominicis and Young 1983) that $\bar{q}$ is equal to $\int_0^1 q(x)\,\mathrm{d}x$ $(=\int_0^1 q(\mathrm{d}x(q)/\mathrm{d}q)\,\mathrm{d}q)$ in the replica symmetry breaking scheme. We notice the correspondence $\bar{P}(q)\to\mathrm{d}x(q)/\mathrm{d}q$.

The above discussion has two steps: a barrier with its height $O(N^{1/2})$ and barriers with height $O(N^{1/4})$. We notice that this discussion is similar to the one of (iii) in § 3 which leads to the Sommers solution. If we want the discussion which leads to the Parisi solution, we have to consider a hierarchical structure with infinite steps such as suggested by the discussion of (iv) in § 3. It should be noticed that a hierarchical structure has been discussed by the replica symmetry breaking scheme (Mézard *et al* 1984).

Here we consider a hierarchical structure with four steps for simplicity. As shown in figure 3, we describe a partial phase space such as $\Omega_{s_1}, \ldots, \Omega_{s_1 s_2 s_3 s_4}$, $s_1, \ldots, s_4 = \pm$ and $P_1(s_1)$, $P_1(s_1)P_2(s_1, s_2), \ldots$ are defined by

$$P_1(s_1) = \mathrm{Tr}_{\Omega_{s_1}} \exp(-\beta H) / \prod_{s_1} \mathrm{Tr}_{\Omega_{s_1}} \exp(-\beta H)$$

$$P_1(s_1) P_2(s_1 s_2) = \mathrm{Tr}_{\Omega_{s_1 s_2}} \exp(-\beta H) / \prod_{s_1} \mathrm{Tr}_{\Omega_{s_1}} \exp(-\beta H)$$

$$\vdots$$

So we have

$$P_1(+) + P_1(-) = 1$$

$$P_2(s_1 +) + P_2(s_1 -) = 1 \tag{47}$$

$$\vdots$$

We describe the local magnetisation in partial phase spaces such as $(m_{is_1}\}$, $\{m_{is_1 s_2}\}, \ldots$. We have the following relations

$$m_{is_1} = \sum_{s_2 = \pm} P_2(s_1 s_2) m_{is_1 s_2}$$

$$m_{is_1 s_2} = \sum_{s_3 = \pm} P_3(s_1 s_2 s_3) m_{is_1 s_2 s_3} \tag{48}$$

$$\vdots$$

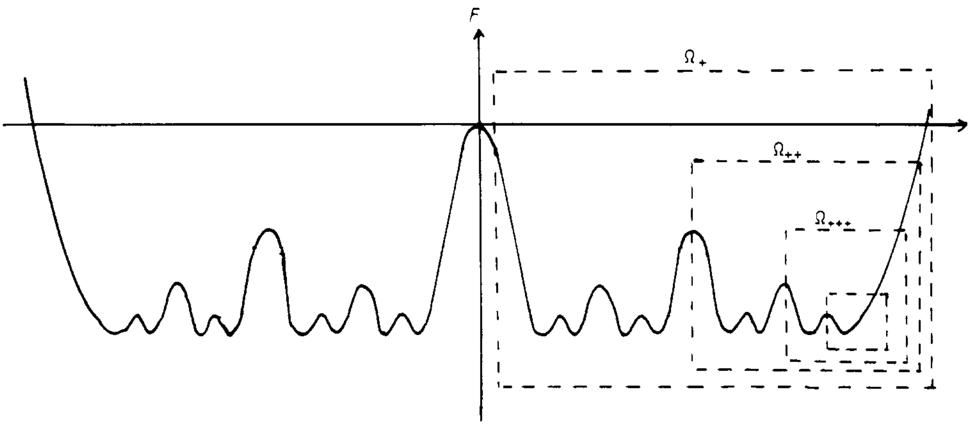

Figure 3. The free energy plotted against phase space in the spin glass phase which has a hierarchical structure with four steps.

4.1. *The case with no symmetry breaking field* ($\delta h = 0$)

We determine the behaviour of $\hat{C}(\omega)$ from the case $\delta h = 0$. The model is symmetric, so we assume

$$P_1(s_1) = P_2(s_1 s_2) = \ldots = \tfrac{1}{2}. \tag{49}$$

We describe the local magnetisation in this case such as $\{\bar{m}_{is_1}\}$, $\{\bar{m}_{is_1s_2}\}, \ldots$. We assume

$$\begin{aligned}&[\bar{m}^2_{is_1s_2s_3s_4}]_a=[\bar{m}^2_{is_1's_2's_3's_4'}]_a\\&[\bar{m}^2_{is_1s_2s_3}]_a=[\bar{m}^2_{is_1's_2's_3'}]_a\\&\quad\vdots\end{aligned}\tag{50}$$

We have the change of the magnitude of freezing

$$[\bar{m}^2_{is_1s_2s_3s_4}]_a \xrightarrow[\varepsilon_4^{-1}]{} [\bar{m}^2_{is_1s_2s_3}]_a \xrightarrow[\varepsilon_3^{-1}]{} [\bar{m}^2_{is_1s_2}]_a \xrightarrow[\varepsilon_2^{-1}]{} [\bar{m}^2_{is_1}]_a \xrightarrow[\varepsilon_1^{-1}]{} 0$$

following the change of the observation timescale $\varepsilon_1^{-1} \gg \varepsilon_2^{-1} \gg \varepsilon_3^{-1} \gg \varepsilon_4^{-1}$. We write $\Sigma_{j=1}^4 q_j=[\bar{m}^2_{is_1s_2s_3s_4}]_a$, $\Sigma_{j=1}^3 q_j=[\bar{m}^2_{is_1s_2s_3}]_a, \ldots$, and we have the correlation function

$$\hat{C}(\omega)=\bar{C}(\omega)+\sum_{j=1}^{4} q_j 2\varepsilon_j(\omega^2+\varepsilon_j^2)^{-1}\tag{51}$$

where $\bar{C}(\omega)$ varies with frequency on the scale of microscopic rates $(T\gamma_0)$.

4.2. *The case with a symmetry breaking field* $(\delta h \simeq O(T/N^{1/2}))$

We describe the local magnetisation and the magnitudes of freezings in a field $\delta h \simeq O(T/N^{1/2})$ such as $\{\hat{m}_{is_1}\}$, $\{\hat{m}_{is_1s_2}\}, \ldots$, and $\hat{q}_1=[\hat{m}^2_{is_1}]_a$, $\Sigma_{j=1}^2 \hat{q}_j=[\hat{m}^2_{is_1s_2}]_a, \ldots$. The symmetry between states is broken, so we assume

$$\begin{cases}P_1(+)=1\\P_1(-)=0\end{cases}\qquad\begin{cases}P_2(s_1+)=p_2\\P_2(s_1-)=1-p_2\end{cases}\qquad\begin{cases}P_3(s_1s_2+)=p_3\\P_3(s_1s_2-)=1-p_3\end{cases}\qquad\begin{cases}P_4(s_1s_2s_3+)=\frac12\\P_4(s_1s_2s_3-)=\frac12\end{cases}$$

$$1>p_2>p_3>\tfrac12.\tag{52}$$

We assume $\hat{m}_{is_1s_2s_3s_4}=\bar{m}_{is_1s_2s_3s_4}$. Using (48) and (52), $\{\hat{m}_{is_1}\}$, $\{\hat{m}_{is_1s_2}\}, \ldots$, are described by $\{\bar{m}_{is_1}\}$, $\{\bar{m}_{is_1s_2}\}, \ldots$. Therefore $\hat{q}_1$, $\Sigma_{j=1}^2 \hat{q}_j, \ldots$, are described by q_1, $\Sigma_{j=1}^2 q_j, \ldots$, as follows:

$$\sum_{j=1}^{4}\hat{q}_j=\sum_{j=1}^{4}q_j \qquad t\ll\varepsilon_4^{-1}\tag{53}$$

$$\sum_{j=1}^{3}\hat{q}_j=\sum_{j=1}^{3}q_j \qquad \varepsilon_4^{-1}\ll t\ll\varepsilon_3^{-1}\tag{54}$$

$$\sum_{j=1}^{2}\hat{q}_j=\sum_{j=1}^{2}q_j+4(p_3-\tfrac12)^2q_3 \qquad \varepsilon_3^{-1}\ll t\ll\varepsilon_2^{-1}\tag{55}$$

$$\begin{aligned}\hat{q}_1=q_1+\sum_{j=2}^{3}4(p_j-\tfrac12)^2q_j-(p_3-\tfrac12)^2p_2(1-p_2)\{[(\bar{m}_{i+++}-\bar{m}_{i++-})-(\bar{m}_{i+-+}-\bar{m}_{i+--})]\}^2]_a\\+2(p_3-\tfrac12)p_2(1-p_2)[(\bar{m}_{i+++}\bar{m}_{i+-+})_a-(\bar{m}_{i++-}\bar{m}_{i+--})_a] \qquad \varepsilon_2^{-1}\ll t.\end{aligned}\tag{56}$$

We assume that $(\bar{m}_{i+++}\bar{m}_{i+-+})_a=(\bar{m}_{i++-}\bar{m}_{i+--})_a$ similar to the assumption (50). Therefore the fourth term on the right-hand side of (56) is zero. The third term on the right-hand side of (56) is the fluctuation term of q_3. This term will become zero when $\hat{N}$, which is the number of steps of a hierarchical structure, becomes infinity. The magnitude of an anomalous response parameter $\hat{\delta}_n$ is given by the difference between the magnitude of freezing in a timescale t_n and that in a timescale t_{n-1} in a field $\delta h \simeq O(T/N^{1/2})$ where $\varepsilon_{j+1}^{-1} \ll t_j \ll \varepsilon_j^{-1}$, as follows:

$$\begin{aligned} T\hat{\delta}_4 &= q_4 \\ T\hat{\delta}_3 &= 4p_3(1-p_3)q_3 \\ T\hat{\delta}_2 &= 4p_2(1-p_2)q_2 + (\text{the fluctuation term of } q_3) \\ T\hat{\delta}_1 &= 0. \end{aligned} \tag{57}$$

We have the response function

$$G(\omega) = \bar{G}(\omega) + \sum_{j=1}^{4} \hat{\delta}_j \varepsilon_j(-\mathrm{i}\omega + \varepsilon_j)^{-1} \tag{58}$$

where $\bar{C}(\omega) = (2T/\omega)\,\mathrm{Im}\,\bar{G}(\omega)$.

We extend the above discussion to the case of a hierarchical structure with $\hat{N}$ steps. If the fluctuation terms of q_j, $j = 1, \ldots, \hat{N}$, can be neglected for a large enough $\hat{N}$, we have the relations

$$T\hat{\delta}_j = 4p_j(1-p_j)q_j \qquad j = 1, \ldots, \hat{N} \tag{59}$$

where

$$\tfrac{1}{2} = p_{\hat{N}} < p_{\hat{N}-1} < \ldots < p_1 = 1. \tag{60}$$

The $4p(1-p)$ changes monotonically from 0 to 1 when p changes monotonically from 1 to $\frac{1}{2}$. Therefore in the limit $\hat{N} \to \infty$, we can put

$$\begin{aligned} &4p_j(1-p_j) \to j/\hat{N} = x_j \\ &\sum_{j=1}^{k} q_j = q(k/\hat{N}) = q(x_k). \end{aligned} \tag{61}$$

We can see that the $4p(1-p)$ corresponds to the Parisi parameter x. The magnitude of freezing in a field $\delta h \simeq O(T/N^{1/2})$ is given by

$$\hat{q}_1 = \sum_{j=1}^{\hat{N}} 4(p_j - \tfrac{1}{2})^2 q_j = \int_0^1 q(x)\,\mathrm{d}x = \int_0^1 q\bar{P}(q)\,\mathrm{d}q \tag{62}$$

in an infinitely long timescale. We conclude from these results that the probability distribution $\bar{P}(q)$ of the overlap q presented by Parisi (1983) is interpreted as the weight function when $\hat{q}_1$ is expressed by using the magnitude of freezing q_j in the symmetric case (figure 4) from the standing point of the dynamical method.

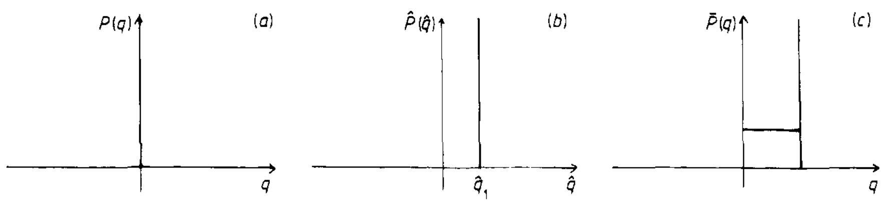

Figure 4. The probability distributions of the overlaps q, $\hat{q}$. (a) The probability distribution $P(q)$ of the overlap q in the symmetric case $\delta h = 0$, (b) the probability distribution $\hat{P}(\hat{q})$ of the overlap $\hat{q}$ in the symmetry breaking case $\delta h = O(T/N^{1/2})$ (see the appendix), (c) the weight function $\bar{P}(q)$ when $\hat{q}_1$ is expressed by using q in the symmetric case $\delta h = 0$.

5. The AC susceptibility of the spin glass

In this section we discuss the spin glass in a finite small field H which is not zero even in the thermodynamic limit $N \to \infty$.

To treat the time dependence of H, we consider physical quantities with no time translational invariance. Substituting (7) into (10), we have

$$\left((1/\gamma_0)\frac{\mathrm{d}}{\mathrm{d}t}+r\right)\sigma_i(t)=\eta_i(t)/\gamma_0+\hat{h}_i(t)+H(t)-\tfrac{1}{3}u\sigma_i^3(t)-\sum_j J_{ij}\sigma_j(t). \tag{63}$$

Using $G_0(t, t')$ defined by

$$\left((1/\gamma_0)\frac{\mathrm{d}}{\mathrm{d}t}+r\right)G_0(t,t')=\delta(t-t') \tag{64}$$

(63) is rewritten as follows:

$$\sigma_i(t)=\int_{-\infty}^{\infty}\mathrm{d}t'\,G_0(t,t')\left[\eta_i(t')/\gamma_0+\hat{h}_i(t')+H(t')-\tfrac{1}{3}u\sigma_i^3(t')-\sum_j J_{ij}\sigma_j(t')\right]. \tag{65}$$

From (65) we can calculate the response function $G(t, t')$ and the correlation function $\hat{C}(t, t')$ in the same approximation as in § 2:

$$\begin{aligned}[\langle\sigma_i(t)\rangle]_{\mathrm{a}}&=\int_{-\infty}^{\infty}G(t,t')\hat{h}_i(t')\,\mathrm{d}t'\\&\quad+(\text{other terms except for the first order terms of }\hat{h})\\G(t,t')&=G_0(t,t')-\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\,G_0(t,t_1)\textstyle\sum(t_1,t_2)G(t_2,t')\end{aligned} \tag{66}$$

$$\begin{aligned}\textstyle\sum(t,t')&=-J^2G(t,t')+\tfrac{3}{2}u\hat{C}(t,t)\delta(t-t')-(9u^2/2)\hat{C}^2(t,t')G(t,t')\\\hat{C}(t,t')&\equiv[\langle\sigma_i(t)\sigma_i(t')\rangle]_{\mathrm{a}}=\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\,G(t,t_1)\hat{\Lambda}(t_1,t_2)G(t',t_2)\\\hat{\Lambda}(t,t')&=H(t)H(t')+(2T/\gamma_0)\delta(t-t')+J^2\hat{C}(t,t')+(3u^2/2)\hat{C}^3(t,t').\end{aligned} \tag{67}$$

We consider that $H(t)$ is an AC field with frequency ω_{a}. For simplicity we assume that $H(t_1)H(t_2)$ is dependent on the difference $t_1 - t_2$ only, as follows:

$$H(t_1)H(t_2)=H^2\cos[\omega_{\mathrm{a}}(t_1-t_2)]. \tag{68}$$

After making this assumption, we can consider that $G(t, t')$ and $\hat{C}(t, t')$ have time translational invariance, i.e. $G(t-t')$, $\hat{C}(t-t')$. Performing the Fourier transform for $t-t'$ in (66) and (67), we have (17)-(19) and

$$\begin{aligned}\hat{\Lambda}(\omega)&=H^2\pi[\delta(\omega-\omega_{\mathrm{a}})+\delta(\omega+\omega_{\mathrm{a}})]+2T/\gamma_0+J^2\hat{C}(\omega)\\&\quad+(3u^2/2)\int_{-\infty}^{\infty}(\mathrm{d}\omega'\,\mathrm{d}\omega''(2\pi)^{-2})\hat{C}(\omega')\hat{C}(\omega'')\hat{C}(\omega-\omega'-\omega'').\end{aligned} \tag{69}$$

With a fixed ω_{a} we can derive the relation equations between $\{q_j\}$ and $\{\hat{\delta}_j\}$ in a similar way to deriving (34). We assume that the other additional relations between $\{q_j\}$ and $\{\hat{\delta}_j\}$ are given by (36) even in a finite external field.

When $\omega_a=0$, H is a static field. In this case only the equation with $n=1$ in (34) is changed, as follows:

$$q_1=G^2(0)[H^2+J^2q_1+(3u^2/2)q_1^3] \tag{70}$$

(for $n=2,\ldots,\hat{N}$, (34) still holds), where $G(0)$ is given by (35) and we consider $\hat{N}$ large enough, so we neglected small quantities $O(1/\hat{N})$. These equations have been obtained in a previous paper (Shirakura 1984a). In that paper the susceptibilities (χ_{FC}, χ_{ZFC}) in a static field were discussed from (70). In the Hertz solution in a static field we consider that the field H induces the freezing q_1, and the random interaction induces the freezing $q_{\hat{N}}$ only. Noticing the Sommers solution to have q_1 and $q_{\hat{N}}$ only, we cannot distinguish between the Hertz solution and the Sommers solution in a static field.

Next we consider that $H(t)$ is an AC field with frequency $\omega_a \simeq \varepsilon_{\hat{N}}$. In this case only the equation with $n=\hat{N}$ in (34) is changed, as follows:

$$q_{\hat{N}}=\left[\left(G(0)-\sum_{j=1}^{\hat{N}}\hat{\delta}_j\right)\left(G(0)-\sum_{j=1}^{\hat{N}-1}\hat{\delta}_j\right)+\hat{\delta}_{\hat{N}}^2/2\right]$$
$$\times\left[H^2+J^2q_{\hat{N}}+\left(\frac{3u^2}{2}\right)q_{\hat{N}}^3+\left(\frac{9u^2}{2}\right)\left(\sum_{j=1}^{\hat{N}-1}q_j\right)q_{\hat{N}}^2+\left(\frac{9u^2}{2}\right)\left(\sum_{j=1}^{\hat{N}-1}q_j\right)^2q_{\hat{N}}\right] \tag{71}$$

(for $n=1,\ldots,\hat{N}-1$, (34) holds).

The magnitude of freezing induced by the external field is non-zero at all temperatures. The phase transition occurs by the freezing of other components. In a static field the field H induces the freezing q_1. The determination equations for q_j, $j=2,\ldots,\hat{N}$, include q_1. Therefore the transition temperature depends on the field H. It was shown (Sompolinsky and Zippelius 1982, Shirakura 1984a) that the transition temperature is given by the de Almeida–Thouless line (de Almeida and Thouless 1978). In an AC field with $\omega_a \simeq \varepsilon_{\hat{N}}$ the finite amplitude H of the field induces the freezing $q_{\hat{N}}$. But the determination equations for q_j, $j=1,\ldots,\hat{N}-1$, do not include $q_{\hat{N}}$. Therefore the transition temperature does not depend on the amplitude H of the AC field.

Next we discuss susceptibilities. The susceptibilities in a static field were discussed in a previous paper (Shirakura 1984a). In that paper we considered that the zero field cooled (ZFC) susceptibility χ_{ZFC} is obtained from the Hertz solution which is stable in a short timescale, and the field cooled susceptibility χ_{FC} is obtained from the Parisi solution which is marginally stable in a long timescale. Here we consider the Hertz solution in an AC field. In the Hertz solution we assumed that the random interaction induces the freezing $q_{\hat{N}}$ only, but now the amplitude H of the AC field with $\omega_a \simeq \varepsilon_{\hat{N}}$ induces the freezing $q_{\hat{N}}$. Therefore we assume that the random interaction induces the freezing $q_{\hat{N}-1}$ only and take the limit $\hat{N}\to\infty$. Then we have the next solution:

$$q_{\hat{N}-1}=[G(\omega_{\hat{N}-2})G(\omega_{\hat{N}-1})+(q_{\hat{N}-1}/T)^2/2][J^2q_{\hat{N}-1}+(3u^2/2)q_{\hat{N}-1}^3]$$
$$q_{\hat{N}}=[G(\omega_{\hat{N}-1})G(\omega_{\hat{N}})+(q_{\hat{N}}/T)^2/2]$$
$$\times[H^2+J^2q_{\hat{N}}+(3u^2/2)q_{\hat{N}}^3+(9u^2/2)q_{\hat{N}-1}q^2+(9u^2/2)q_{\hat{N}-1}^2q] \tag{72}$$

$$G(\omega_{\hat{N}-2})=1/T \qquad G(\omega_{\hat{N}-1})=(1-q_{\hat{N}-1})/T \qquad G(\omega_{\hat{N}})=(1-q_{\hat{N}-1}-q_{\hat{N}})/T.$$

We consider that the AC susceptibility with a finite amplitude H is given by $G(\omega_{\hat{N}})$, i.e. $\chi_{AC} = G(\omega_{\hat{N}})$. For simplicity we put $J = 1$. The transition temperature T_c is given by $T_c = 1$. Near the transition temperature, χ_{AC} is given by

$$\begin{aligned} \chi_{AC} \equiv G(\omega_{\hat{N}}) &\simeq 1 + \tau - H^2/2|\tau| && \tau \ll -H \\ &\simeq 1 - H && |\tau| \ll H \\ &\simeq 1 - \tau - H^2/2\tau && \tau \gg H \end{aligned} \tag{73}$$

where $\tau \equiv 1 - T$. χ_{AC} is shown in figure 5 with χ_{ZFC} and χ_{FC} (Shirakura 1984a) for comparison. $\chi_{AC}(H \neq 0)$ is rounded inside the cusp of the $\chi_{AC}(H = 0) = \chi_{ZFC}(H = 0)$. This result is often observed in many experiments.

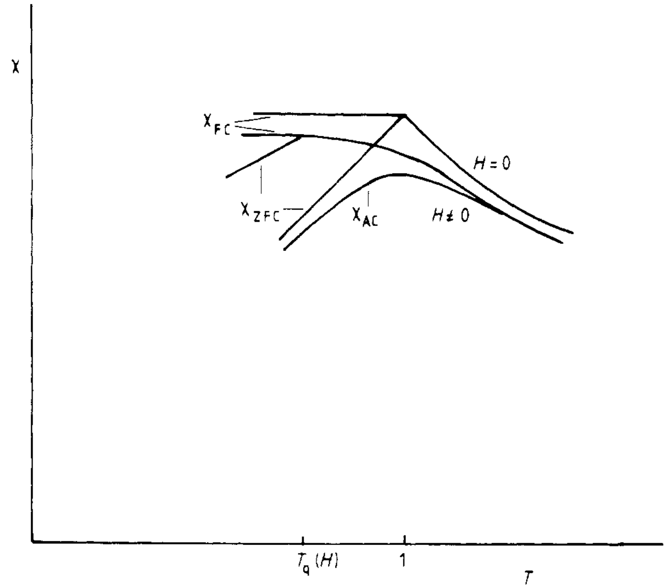

Figure 5. The plots of susceptibilities against temperature at $H = 0$ and $H \neq 0$. χ_{ZFC} and χ_{FC} are the ZFC and FC susceptibilities, respectively (Shirakura 1984a) and χ_{AC} is the AC susceptibility.

We should notice that all the ε_j become zero in the thermodynamic limit $N \to \infty$. Therefore the theory in this paper can not give the real timescales in experiments. But the results in this section suggest that the reason the temperature dependences of AC susceptibilities in SG materials are rounded inside the cusp as the amplitude H of the AC field is increased is explained by the result that the AC field induces the freezing of clusters with the same relaxation time as the cycle time of the AC field, and the reason that the cusps of the zero field cooled susceptibilities in SG materials are shifted to lower temperatures as the static field H is increased is explained by the result that the increase of the static field H shifts the transition temperature to lower temperatures. On the other hand, Fischer (1983) and Fischer and Kinzel (1984) discussed the AC susceptibilities with an infinitesimally small amplitude in a static external field H and presented interesting results.

6. Discussion

In the previous section we assumed that χ_{ZFC} and χ_{AC} are obtained from the Hertz solutions and χ_{FC} is obtained from the Parisi solution. Some papers insist that χ_{ZFC} is obtained from the case $x=1$ in the Parisi solution and χ_{FC} is obtained from the case $x=0$ in the Parisi solution, but we do not think so. We consider that the difference between the cases $x=1$ and $x=0$ in the Parisi solution is the difference between the observation timescales, but the difference between χ_{ZFC} and χ_{FC} is the difference between the observation times from the put-on time of the field H after zero field cooling. Therefore this is a non-equilibrium phenomenon. At a short time from the put-on time when we observe χ_{ZFC}, all spin clusters behave dynamically to proceed to a new equilibrium state. Therefore we have the Hertz solution in this case. At a long time from the put-on time the system becomes an equilibrium state in the field H. In this case we have the Parisi solution, and we can select the case $x = 0$–1 following the observation timescale.

Next we discuss the free energy structure in a static external field H similar to the one in § 4. We assume that the symmetry breaking field δh which is static, uniform and infinitesimally small in the limit $N \to \infty$ changes the mixing probabilities between states and the static uniform external field H changes the free energy structure itself, and we can distinguish between the symmetric case ($\delta h = 0$) and the symmetry breaking case ($\delta h \simeq \mathrm{O}(T/N^{1/2})$) even in the field H below the de Almeida–Thouless line. In a small field H we consider the similar free energy structure to the one in § 4. We consider that the symmetric case on the field H has the next mixing probability $\{\bar{p}_j\}$.

$$\begin{aligned} \bar{p}_j &= 1 \qquad && j = 1, \dots, n \\ &= \tfrac{1}{2} && j = n+1, \dots, k \end{aligned}$$

where we assumed the hierarchical structure with k steps. We consider that the symmetry breaking case in the field H has the following mixing probabilities $\{\hat{p}_j\}$:

$$\begin{aligned} \hat{p}_j &= 1 \qquad && j = 1, \dots, n \\ &= p_j && j = n+1, \dots, k \end{aligned}$$

where p_j is related to the Parisi parameter x_j through (61), and if the PT hypothesis holds (Parisi and Toulouse 1980), n depends only on the field H and k depends only on the temperature T. Thus we can repeat the discussion in a similar way to § 4.

Finally we discuss the symmetry breaking field δh. In this paper we consider $\delta h \simeq \mathrm{O}(T/N^{1/2})$, but we may probably have some arbitrariness of the magnitude of δh. We think that this arbitrariness may be related to the arbitrariness of $f(x)$ where $f(x)$ is defined by the additional relation equation between $\hat{\delta}(x)$ and $q(x)$, $T\hat{\delta}(x) = f(x)q(x)$. In the previous sections we fix $f(x) = x$, but $f(x)$ can be some arbitrary function which is continuous and monotonously increasing, and satisfies $f(x=1)=1$ and $f(x=0)$ (Sompolinsky 1981, Horner 1984a, b).

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Appendix

The probability distribution $P(q)$ of the overlap q in the symmetric case $\delta h = 0$ and the probability distribution $\hat{P}(\hat{q})$ of the overlap $\hat{q}$ in the symmetry breaking case $\delta h \simeq O(T/N^{1/2})$ is found.

We consider the probability distributions $P(q)$, $\hat{P}(\hat{q})$ of the overlaps q, $\hat{q}$ of the magnetisation between two states in the symmetric case $\delta h = 0$ and in the symmetry breaking case $\delta h \simeq O(T/N^{1/2})$. We consider $Y_q \equiv \int_q^1 \mathrm{d}q' \, P(q')$ and $\hat{Y}_{\hat{q}} \equiv \int_{\hat{q}}^1 \mathrm{d}\hat{q}' \, \hat{P}(\hat{q}')$. We can easily calculate Y_q and $\hat{Y}_{\hat{q}}$ such as

$$Y_{\Sigma_{j=1}^k q_j} = \tfrac{1}{2}^k$$

$$\hat{Y}_{\Sigma_{j=1}^k \hat{q}_j} = \prod_{j=1}^{k} [p_j^2 + (1-p_j)^2] = \prod_{j=1}^{k} [1 - 2p_j(1-p_j)].$$

In the continuum limit $\hat{N} \to \infty$, we have $P(q) = \delta(q)$ and $\hat{P}(\hat{q}) = \delta(\hat{q} - \hat{q}_1)$, because if $x \neq 0$, we have $Y_{q(x)} = \Pi_{j=1}^{\hat{N}x} (\frac{1}{2}) = 0$ and $\hat{Y}_{\hat{q}(x)} = \Pi_{j=1}^{\hat{N}x} (1 - j/2\hat{N}) = 0$.

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